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NOTE

Note on Certain Inequality

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Using only the elementary properties of lattice-ordered groups, we give a simple proof of the inequality of Maligranda and Orlicz [2] in full generality.

INTRODUCTION

Using the integral representation of a convex function, Maligranda and Orlicz [2, 3.3] have proved the following inequality:

$$\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| \le f\left(\sum_{k=1}^{n} |a_k - a_{k-1}| \right)$$

(f: $[0, \beta\rangle \to [0, \infty\rangle$ being a convex function such that $f(0) = 0$; a_k being nonnegative reals such that $a_0 = 0$). This result is generalized in [3] on functions f: $[0, \beta_1\rangle \times \cdots \times [0, \beta_r\rangle \to [0, \infty\rangle$ with increasing increments. The proof in [3] depends on the fact that the image of the function f, $\operatorname{Im} f$, is a subset of $[0, \infty\rangle$ (the fact that $\operatorname{Im} f$ is a linearly ordered set has essentially been used).

In this short note a simple proof of the inequality in the general situation and in a stronger form (for functions with increasing increments on lattice-ordered groups) is presented (Theorem 1). Also, we give a version of the inequality for increasing functions with decreasing increments (Theorem 2).

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1. LATTICE-ORDERED GROUPS

A lattice-ordered group [1, Chap. VI, Sects. 8 and 9] is an ordered group G such that there exist $\sup(x, y)$ and $\inf(x, y)$ for every $x, y \in G$. Note that $\inf(x, y) = -\sup(-x, -y)$ for every $x, y \in G$. It can easily be seen that, for an increasing function $f\colon G \to H$ from lattice-ordered group G to lattice-ordered group H,

$$\sup\{f(a), f(b)\} \leq f(\sup\{a, b\}) \tag{1}$$

and

$$\inf\{f(a), f(b)\} \geq f(\inf\{a, b\}). \tag{2}$$

Let us define a G-norm $\|$ on G [1, Chap. VI, Definition 4]

$$\|\colon G \to G^{+} = \{a \in G\colon a \geq 0\}, \qquad |a| = \sup\{a, 0\} - \inf\{a, 0\}.$$

We have [1, Chap. VI, Prop. 9]

(i) $|a| \geq 0$;

(ii) $|a| = 0$ if and only if $a = 0$;

(iii) $|a + b| \leq |a| + |b|$ (triangle inequality);

(iv) $|-a| = |a|$.

For instance, (iii) follows from the obvious inequalities

$$\sup\{a + b, 0\} \leq \sup\{a, 0\} + \sup\{b, 0\}$$

and

$$\inf\{a + b, 0\} \geq \inf\{a, 0\} + \inf\{b, 0\}.$$

Also, we have: If $a \geq 0$, then $|a| = a$; further, if $a \leq 0$, then $|a| = -a$.

The functions sup and inf satisfy the additional property:

$$\sup\{a + c, b + c\} - \inf\{a + c, b + c\} = \sup\{a, b\} - \inf\{a, b\}. \tag{3}$$

Namely, $\sup\{a + c, b + c\} = \sup\{a, b\} + c$ and $\inf\{a + c, b + c\} = \inf\{a, b\} + c$. Directly from (3) we obtain (for every $a, b \in G$)

$$|a - b| = \sup\{a, b\} - \inf\{a, b\}. \tag{4}$$

EXAMPLE 1. Let $G = \mathbb{R}^k$, $k \in \mathbb{N}$. For $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$, define $a \leq b$ if we have $\forall i$, $a_i \leq b_i$. Then, $\mathbb{R}^k$ is a lattice-ordered group. If $k = 1$, then G is the field of reals with standard ordering and $\|$ is the standard absolute value.

2. FUNCTIONS WITH INCREASING (DECREASING) INCREMENTS

Let G, H be lattice-ordered groups. Then we say that $f\colon G^+ \to H^+$ is a function with increasing increments if

$$f(a+h) - f(a) \leq f(b+h) - f(b), \tag{5}$$

whenever $h \geq 0$ and $a \leq b$. If the equality in (5) is satisfied if and only if $a = b$ or $h = 0$, then we say that f is a function with strictly increasing increments. We say that f is a function with decreasing increments if the reverse inequality in (5) holds. Analogously, we obtain strictly decreasing increments.

Putting $a = 0$ in (5), we see that a function $f\colon G^+ \to H^+$ with increasing increments such that $f(0) = 0$ in an increasing function.

EXAMPLE 2. Consider the group $\mathbb{Z}$ with standard relation $\leq$. Define $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ by $f(n) = n^2$ for $n \neq 1$ and $f(1) = 0$. Then f is a function with strictly increasing increments, but f is not a strictly increasing function. We will prove this by analyzing the inequality $f(a+h) - f(a) \leq f(b+h) - f(b)$. If $h = 0$ or $a = b$, we have the equality. Suppose that $h > 0$ and $a < b$. Then:

(a) if $a = 0$ and $h = 1$, we have $0 < f(b+1) - f(b)$;

(b) if $a = 0$ and $h > 1$, we have $h^2 < (b+h)^2 - b^2$, that is $0 < 2bh$;

(c) if $a = 1$, we have $(h+1)^2 < (b+h)^2 - b^2$, that is $1 < 2h(b-1)$;

(d) if $a > 1$, we have $(a+h)^2 - a^2 < (b+h)^2 - b^2$, that is $a < b$.

Remark. If G is a lattice-ordered group divisible by $n \in \mathbb{N}$, $n \neq 1$, and if $a \geq 0$ implies $a/n \geq 0$ for every $a \in G$, then one can prove that a function $f\colon G^+ \to G^+$ with strictly increasing increments such that $f(0) = 0$ is a strictly increasing function.

LEMMA 1. *Let $f\colon G^+ \to H^+$ be a function with increasing increments such that $f(0) = 0$. Assume that $a, b, c \in G^+$ such that $a \geq c$. Then*

$$f(a) + |f(b) - f(c)| \leq f(a + |b-c|). \tag{6}$$

If f is a function with strictly increasing increments, then the equality in (6) *holds if and only if $b = c$* or *$a = c \leq b$.*

Proof.

$$\begin{aligned} f(a) + |f(b) - f(c)| &= f(a) + \sup\{f(b), f(c)\} - \inf\{f(b), f(c)\} \\ &\leq f(a) + f(\sup\{b, c\}) - f(\inf\{b, c\}) \\ &\leq f(a + \sup\{b, c\} - \inf\{b, c\}) = f(a + |b - c|) \end{aligned}$$

(see (1), (2), and (4) and recall that f must be an increasing function).

If f is a function with strictly increasing increments, then the second inequality in the proof of the first part of Lemma 1 becomes the equality if and only if $\sup\{b, c\} = \inf\{b, c\}$ (i.e., if $b = c$) or if $a = \inf\{b, c\}$ (i.e., if $a \leq c$, hence $a = c \leq b$). One can see then that the first inequality becomes the equality, too. Therefore, the equality in (6) holds if and only if $b = c$ or $a = c \leq b$. ∎

THEOREM 1. *Let f be as in Lemma* 1 *and let* $0 = a_0, a_1, \ldots, a_n \in G^+$. *Then*

$$\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| \leq f\left(\sum_{k=1}^{n} |a_k - a_{k-1}| \right). \tag{7}$$

If f is a function with strictly increasing increments, then the equality in (7) *holds if and only if* $a_1 \leq \cdots \leq a_n$.

Proof. If $n = 1$, we have the equality. Let $n = m - 1$. Denote $a = \sum_{k=1}^{n} |a_k - a_{k-1}|$. From the triangle inequality we see that $a \geq a_{m-1}$. By the assumption of induction, we reduce all to the inequality

$$f(a) + |f(a_m) - f(a_{m-1})| \leq f(a + |a_m - a_{m-1}|), \tag{8}$$

which easily follows by (6).

If f is a function with strictly increasing increments, then, by Lemma 1, the equality in (8) holds if and only if $a_m = a_{m-1}$ or $a = a_{m-1} \leq a_m$. By the proof of the first part of Theorem 1, we obtain

$$\sum_{k=1}^{m} |f(a_k) - f(a_{k-1})| \leq f(a) + |f(a_m) - f(a_{m-1})| \leq f(a + |a_m - a_{m-1}|). \tag{9}$$

If $a_m = a_{m-1}$ or $a = a_{m-1} \leq a_m$, then the first inequality in (9) becomes the equality if and only if $\sum_{k=1}^{m-1} |f(a_k) - f(a_{k-1})| = f(\sum_{k=1}^{m-1} |a_k - a_{k-1}|)$. Hence, we conclude by the assumption of induction. ∎

The analogue of Theorem 1 does not generally hold for functions with decreasing increments (see [3, the end of Sect. 2]). We will prove a version of Theorem 1 for such functions in the case that $a_1, \dots, a_n$ are comparable (i.e., if for every i, j we have $a_i \leq a_j$ or $a_i \geq a_j$). This condition is automatically satisfied if the group G is linearly ordered.

LEMMA 2. *Let f: $G^+ \to H^+$ be an increasing function with decreasing increments such that* $f(0) = 0$. If *a, b, c are comparable (especially if G is linearly ordered) and if $a \geq c$, then*

$$f(a) + |f(b) - f(c)| \geq f(a + |b - c|). \tag{10}$$

If f is an increasing function with strictly decreasing increments, then the equality in (10) *holds if and only if $b = c$* or *$a = c \leq b$.*

Proof. Let $b \geq c$. Then (10) becomes $f(a) + f(b) - f(c) \geq f(a + b - c)$, that is, $f(a) - f(c) \geq f(a + b - c) - f(b)$. This inequality follows by the definition (recall that $a \geq c$). Let $b \leq c$. Then (10) becomes $f(a) - f(b) + f(c) \geq f(a + c - b)$, that is, $f(a) - f(b) \geq f(a + c - b) - f(c)$.

If f is an increasing function with strictly decreasing increments, then the equality in the first case holds if and only if $a = c \leq b$ or $b = c$. The equality in the second case holds if and only if $b = c$ or $a = b$ (hence $a = b = c$). Therefore, the equality in (10) holds if an only if $b = c$ or $a = c \leq b$. ∎

THEOREM 2. *Let f be as in Lemma* 2 *and let* $0 = a_0, a_1, \dots, a_n \in G^+$ *be comparable (especially let G be linearly ordered). Then*

$$\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| \geq f\left(\sum_{k=1}^{n} |a_k - a_{k-1}| \right). \tag{11}$$

If f is an increasing function with strictly decreasing increments, then the equality in (11) *holds if and only if $a_1 \leq \cdots \leq a_n$.*

Proof. If $n = 1$ we have the equality. Let $n = m - 1$. Denote $a = \sum_{k=1}^{n} |a_k - a_{k-1}|$. By the assumption of induction, we reduce the problem on the inequality:

$$f(a) + |f(a_m) - f(a_{m-1})| \geq f(a + |a_m - a_{m-1}|), \tag{12}$$

which easily follows by Lemma 2.

If f is a function with strictly decreasing increments, then the equality in (12) holds if and only if $a = a_{m-1}$ or $a_m = a_{m-1}$. Hence, the result follows by induction (as in the proof of Theorem 1). ∎

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